6-1-1985

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Recommended Citation
Kim, Y., & Harris, A. (1985). Density of States of the Random-Hopping Model on a Cayley Tree. *Physical Review B, 31* (11), 7393-7407. http://dx.doi.org/10.1103/PhysRevB.31.7393

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Abstract
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Disciplines
Physics | Quantum Physics

Density of states of the random-hopping model on a Cayley tree

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(Received 6 October 1983; revised manuscript received 15 June 1984)

We formulate an integral equation and recursion relations for the configurationally averaged one-particle Green's function of the random-hopping model on a Cayley tree of coordination number $\sigma+1$. This formalism is tested by applying it successfully to the nonrandom model. Using this scheme for $1 \ll \sigma < \infty$, we calculate the density of states of this model with a Gaussian distribution of hopping matrix elements in the energy range $E^2 > E_c^2$, where E_c is a critical energy described below. The singularity in the Green's function which occurs at energy $E_1^{(0)}$ for $\sigma = \infty$ is shifted to complex energy E_1 (on the unphysical sheet of energy E) for small σ^{-1}. This calculation shows that the density of states is a smooth function of energy E around the critical energy $E_c = \mathrm{Re}E_1$, in accord with Wegner's theorem. In this formulation the density of states has no sharp phase transition on the real axis of E because E_1 has developed an imaginary part. Using the Lifschitz argument, we calculate the density of states near the band edge for the model when the hopping matrix elements are governed by a bounded probability distribution. This case is also analyzed via a mapping similar to those used for dynamical systems, whereby the formation of energy band can be understood.

I. INTRODUCTION

Since Anderson's original work,[1] there has been much interest[2-19] in characterizing the nature of the eigenstates of a quantum-mechanical particle in a random potential. For the purposes of the present paper we will limit our consideration to models of noninteracting electrons (or other excitations) on discrete lattices described by a randomized nearest-neighbor tight-binding model whose Hamiltonian is

$$H = \sum_i V_i c_i^\dagger c_j + \sum_{\langle i,j \rangle} t_{ij}(c_i^\dagger c_j + c_j^\dagger c_i) , \tag{1.1}$$

where $\langle i,j \rangle$ denotes a sum over nearest-neighbor bonds, and $c_i^\dagger$ (c_i) creates (destroys) an excitation on site i. All the variables $\{V_i\}$ and $\{t_{ij}\}$ are uncorrelated random variables with a probability distribution P,

$$P(\{V_i\},\{t_{ij}\}) = \prod_i p_V(V_i) \prod_{\langle i,j \rangle} p(t_{ij}) . \tag{1.2}$$

Averages with respect to P will be indicated by $[\ ]_P$. When there is no randomness, i.e., for $p_V = \delta(V)$ and $p(t_{ij}) = \delta(t_{ij} - t_0)$, the eigenstates of H are plane waves, i.e., extended states. When either $p_V(V)$ or $p(t_{ij})$ has large dispersion, so that either $[V_i^2]_p - [V_i]_p^2 \gg [t_{ij}]_p^2$ or $[t_{i,j}^2]_p - [t_{i,j}]_p^2 \gg [t_{i,j}]_p^2$, then many, if not all, states will be localized by the disorder. In the case when both localized and extended states occur, the situation can be represented as in Fig. 1(a), where we indicate that states with energy near the center of the band are extended, whereas states far from the center of the band are local-

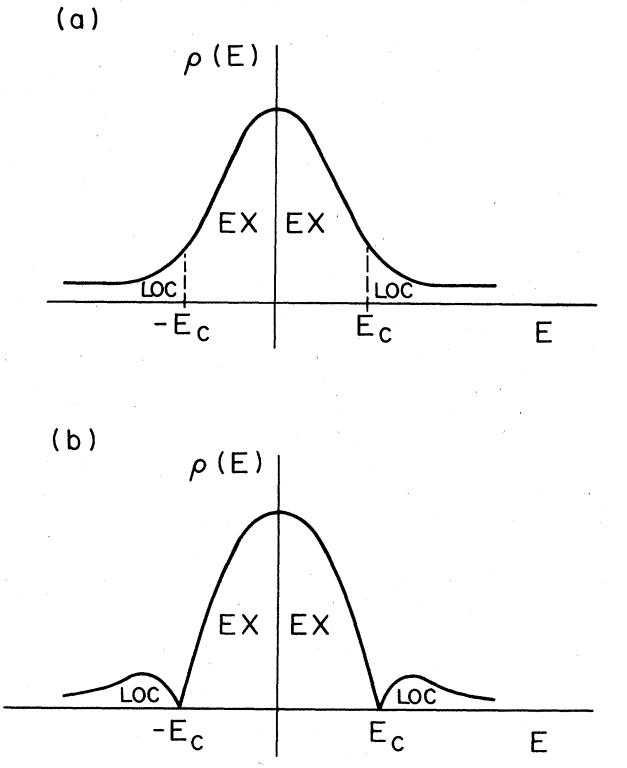

FIG. 1. (a) Conventional view of density of states for bond random model. Here, "EX" denotes extended states and "LOC" denotes localized states. (b) An alternative view of density of states for bond random model.

ized. The behavior of states around the critical energy (mobility edge) E_c (or $-E_c$) has been discussed[6-19] in the context of critical phenomena and is the principal object of our study. The conventional view[5] of the densities of extended and localized states in the general Anderson model is expressed in Fig. 1(a). The density of states is usually assumed to decrease smoothly into the band-tail regions, crossing the mobility edge without any perceptible change of structure at the mobility edge. This view is supported by an exact solution for the case when the t_{ij}'s are not randomized but the V_i's are governed by a Lorentzian distribution. In this case, as shown by Lloyd,[14] the density of states is analytic despite the fact that, presumably, for sufficiently large randomness, localized states are expected to appear at large energies. In fact, a theorem due to Edwards and Thouless[8] shows that the density of states is analytic when there is only site randomness present. This result was extended by Wegner,[19] who has given a general proof, for bounded probability distributions of the t's or V's, that the density of states neither vanishes nor diverges within the band, except possibly at zero energy. Since the mobility edge does not, in general, occur at $E=0$, Wegner's work shows that the density of states is analytic and nonzero at the mobility edge and, therefore, that the single-particle Green's function is probably analytic (in E) at the mobility edge.

Attempts to construct a field-theoretic formulation analogous to those for ordinary thermodynamic critical phenomena have been successful only for low spatial dimensionality d, i.e., for d near 2.[9,10] In analogy with critical phenomena, one might have anticipated the development of theories in which mean-field theory formed the starting approximation, with fluctuation becoming relevant for $d<d_c$, where d_c is the so-called upper critical dimension. Such a program has not been successfully carried out, and, in fact, several guesses for the value of d_c have appeared in the literature.[13,15,16]

The most complete field-theoretic formulation for large d given so far is that of Harris and Lubensky.[15,16] In their work the effective Hamiltonian for the Gaussian bond model with $[t_{ij}]_p=0$ and $[t_{ij}^2]_p\neq 0$ takes the form

$$H_{\rm eff}=\frac{1}{4[t_{ij}^2]}\sum_{\alpha,\beta,\theta,\theta',x,x'} Q_{\alpha\beta}^{\theta\theta'}(x)[\gamma^{-1}]_{xx'}Q_{\alpha\beta}^{\theta\theta'}(x') + \tfrac{1}{2}\sum_{x,\alpha,\theta}\ln(E+i\theta\eta+Q_{\alpha\beta}^{\theta\theta}) , \tag{1.3}$$

where α runs from 1 to n, with the limit $n\to 0$ implied, θ and θ' are ± 1, γ is a matrix in the position labels x and x' which vanishes unless x and x' are nearest neighbors, in which case it is unity, and $Q_{\alpha\beta}^{\theta\theta'}(x)$ is a unitary matrix (considering $\alpha\theta$ as one index and $\beta\theta'$ as another). The averaged single-particle Green's function $[G(x,x,E)]_p$ is given by

$$[G(x,x;E+i\eta)]_p=\langle Q_{\alpha\alpha}^{11}(x)\rangle_{H_{\rm eff}}\equiv Q , \tag{1.4}$$

and the averaged two-particle Green's function is given by

$$[G(x,x';E+i\eta)G(x,x';E-i\eta)]_p = \langle Q_{\alpha\beta}^{1}{}^{-1}(x)Q_{\alpha\beta}^{1}{}^{-1}(x')\rangle_{H_{\rm eff}} , \tag{1.5}$$

where η is a positive infinitesimal. Here we defined

$$\langle A\rangle_H\equiv \mathrm{Tr}A\exp(-H_{\rm eff})/\mathrm{Tr}\exp(-H_{\rm eff}) ,$$

where Tr indicates an integration over all Q's.

In this theory, Q has an imaginary part wherever there is a nonzero averaged density of states, since

$$\rho(E,x)=-(1/\pi)\,\mathrm{Im}[G(x,x;E+i\eta)]_p . \tag{1.6}$$

In mean-field theory, i.e., when all nonlocal effects in Eq. (1.4) are ignored, minimization of $H_{\rm eff}$ yields a self-consistent equation for Q,

$$\frac{1}{z[t^2]_p}Q+\frac{1}{Q+E}=0 , \tag{1.7a}$$

whose solution is

$$Q=\frac{E}{2}\left[\left[1-\frac{4z[t^2]_p}{E^2}\right]^{1/2}-1\right] , \tag{1.7b}$$

where z is the coordination number of the lattice. In mean-field theory, then, there are states only in the region $-E_c^{\rm MF}<E<E_c^{\rm MF}$, where $E_c^{\rm MF}$ is the mean-field value for the mobility energy: $E_c^{\rm MF}=2(z[t^2]_p)^{1/2}$. When fluctuations away from the mean-field solution are considered, the conclusions are as follows.

(1) Extended states are described by perturbative fluctuations about the mean-field minimum of the effective Hamiltonian.

(2) These perturbative fluctuations can be treated by renormalization-group techniques and they are described by a fixed point which is non-mean-field-like in less than eight spatial dimensions.

(3) Localized states in this theory are described by instanton solutions. Naively speaking, for an infinite number of integration variables, these are the generalizations of saddle-point contributions to an integral in a single variable.

(4) No coupling was found between the perturbative fluctuations and the instanton solutions. Consequently, the field theory of Ref. 16 led to the result that there was a singularity at the mobility edge where the density of states vanished, as in Fig. 1(b), according to

$$\rho(E)\propto |E-E_c|^{\beta} , \tag{1.8}$$

with a characteristic exponent β which was not considered to take a conventional (or van Hove) value of $\beta=d/2-1$. It seems possible to introduce the exponent β for the conventional model [as in Fig. 1(a)] if $\rho(E)$ in Eq. (1.8) is interpreted to be the density of *extended* states. A density of states such as that of Fig. 1(b) has also been predicted for the three-dimensional Anderson model by Haydock,[7] who used a continued-fraction formalism and obtained results which are claimed to be exact to lowest order in the scattering strength. Haydock found $\beta=d/2-1$, but made no specific prediction for the form of the density of localized states.

However, such a singularity in the density of states is surely spurious in view of Wegner's theorem.[19] While that theorem has not been proved for an unbounded distribution such as a Gaussian, it seems implausible that the calculations are sensitive to whether or not the Gaussian

distribution is truncated at some high energy. A natural question, which has so far remained unanswered, is where in the field theory is the coupling between perturbative fluctuations and instantons. It is clear from the field theory that such a coupling is relevant (in the renormalization-group sense) and, hence, that it will lead to a smooth density of states, in accord with Wegner's theorem.[19] Such a smooth density of states has been obtained in several works of Ziegler *et al.*[18] However, it should be stressed that they simply impose, from the outset, a coupling, by stating that $\langle Q \rangle$ has a nonzero imaginary part at the mobility edge. They do not discuss the origin of this term from an effective Hamiltonian such as that of Eq. (1.5), other than to note that the nonvanishing $\mathrm{Im}\langle Q \rangle$ is required to give a nonzero density of states.

In summary, we emphasize that because this coupling between perturbative fluctuations and instantons is not understood, a satisfactory mean-field theory of localization in high spatial dimensions has yet to be given. Since this elusive coupling is difficult even to locate, much less to treat, in a field theory, we have decided to study the bond model on a Cayley tree, where the results can be expressed in terms of the solution to a nonlinear integral equation, and thus we eliminate difficulties introduced by the infinite number of degrees of freedom present in any field theory. (A section of the Cayley tree of coordination number 3 is shown in Fig. 2.) As can be seen from the ordinary second-order phase-transition problems,[20] the Cayley tree corresponds to the infinite-dimensional hypercubic lattice and, hence, should be described by mean-field theory.

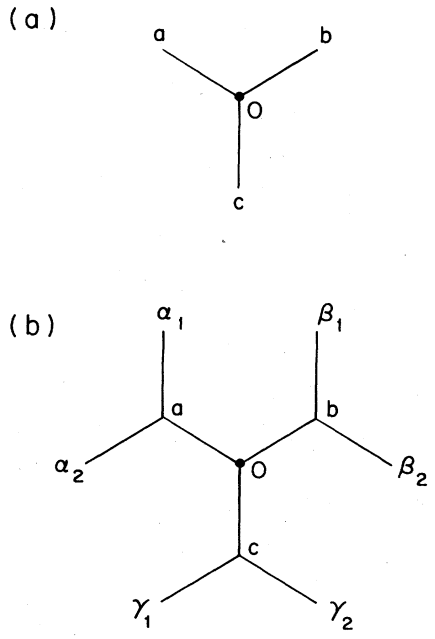

FIG. 2. (a) 1-generation tree of coordination number $z=\sigma+1=3$, whose origin is at "0." (b) 2-generation tree of coordination number $z=\sigma+1=3$, whose origin is at "0."

Our treatment of this problem is based on the replica trick.[21–23] We formulate an integral equation and recursion relations for the configurationally averaged one-particle Green's function $[G(x,x;E+i\eta)]_p$, from which one obtains the configurationally averaged density of states at site x via Eq. (1.6). Our procedure is as follows: First, we reproduce the well-known mean-field-theory result,[16] i.e., Eqs. (1.7), in the limit $z\to\infty$, where z is the coordination number of the Cayley tree. Next, we analyze the correction due to finite z. There are two kinds of corrections. One is perturbative in $1/\sigma$, where $z=\sigma+1$. This is a somewhat trivial effect and does not have any physically important consequence, as can be seen from the ordinary phase-transition problem[24] on a Cayley tree. The other one is peculiar to this model and involves the above-mentioned coupling to instantons. As we shall see, this coupling explains why there is no sharp transition as far as $[G(x,x';E+i\eta)]_p$ and $\rho(E,x)$ are concerned, in agreement with Wegner's theorem. This coupling leads to a small imaginary contribution of order γ to the equation of state, i.e., the self-consistent equation for Q analogous to Eq. (1.7a), where $\gamma\sim iae^{-b\sigma}$, where a and b are unimportant constants. As a consequence, the critical point is shifted to the complex energy (on the unphysical sheet of the Green's function) $E_1=E_{c,p}-i\gamma$, where $E_{c,p}$ includes the perturbative contribution to E_c. Although physical energies can never reach this singularity, at least for large σ the energy dependence of quantities near E_c is still dominated by the nearby critical point at complex energy. Hopefully, our results will lead to the discovery of the coupling between perturbative and instanton effects in the field theory which reproduces this effect and thereby brings the ϵ-expansion treatment into accord with Wegner's theorem.

Recently, Kunz and Souillard[25] have shown rigorously that on a Cayley tree the model with only site-diagonal randomness gives results similar to ours. However, our work is the first to explicitly analyze the coupling between perturbative fluctuations and instantons, which are essential to a coherent theory of localization.

Briefly, this paper is organized as follows. In Sec. II we formulate the integral equation for the Green's function on the Cayley tree using the replica method, and we check this formulation by applying it to the nonrandom case in which all the hopping constants are the same. In addition, the $1/\sigma$ expansion of the density of states is developed and we show that this expansion satisfies the sum rule on the density of states. In Sec. III the density of states of this model with a Gaussian distribution is calculated through the steepest-descent integral method, and it is shown that the shift of the critical energy E_1 to the unphysical sheet explains why there is no sharp transition on the real axis. We also show that, near $E=\mathrm{Re}E_1\equiv E_c$, the dominant contribution to the density of states comes from the coupling between the perturbative $1/\sigma$ expansion and the nonperturbative saddle-point contribution, while for $E\gg E_c$ the dominant contribution comes from the saddle-point integral only. Thus there is a crossover from the high-energy regime, where the saddle-point contributions dominate, to the "critical regime," where the cou-

pling between the perturbative and nonperturbative methods is dominant. In Sec. IV, using a recursion-relation formulation and the Lifschitz argument,[26] we calculate the density of states near the band edge for the finite-distribution case. Furthermore, it is explained why the density of states never vanishes inside the band using the language of dynamical-system maps.[27–29] In Sec. V the final discussions and conclusions are given.

II. FORMULATION OF INTEGRAL EQUATION FOR THE CAYLEY TREE

A. Green's function and integral equation

In this section we derive the integral equation for the Green's function on the Cayley tree using the replica method.[21–23] The field-theoretic formulation[16] for Green's function $iG(0,0;E+i\eta)$ of the random hopping model of Eq. (1.1) with $p_V(V_i)=\delta(V_i)$, i.e., bond-random model, can be written as

$$i[G(0,0;E+i\eta)]_t=\int x_{0\alpha}^2 \exp\left[\frac{i}{2}\left[(E+i\eta)\sum_{i,\alpha}x_{i\alpha}^2\right]\right]\prod_b F(x_{i\alpha},x_{j\alpha})\prod_{i,\alpha}dx_{i\alpha}\ , \tag{2.1}$$

with

$$F(x_{i\alpha},x_{j\alpha})=\left[\exp\left[-it_{ij}\sum_{i,\alpha}x_{i\alpha}x_{j\alpha}\right]\right]_t \equiv F(\vec{x}_i,\vec{x}_j)\ , \tag{2.2}$$

where $[\ ]_t$ indicates an average over the probability distribution of the t_{ij}'s, and $\prod_b$ denotes a product over bonds. Also, the indices i and j are the site labels, "0" means the center site of the Cayley tree, and α is the replica index, where $\alpha=1,2,\ldots,n$, and the limit $n\to 0$ is always implied. To simplify the notation, we define

$$g\equiv\lim_{\eta\to 0}[G(0,0;E+i\eta)]_t\ . \tag{2.3}$$

Now consider $\vec{x}_i$ an n-component vector. Then,

$$ig=\frac{1}{n}\int x_0^2\exp\left[\frac{iE}{2}\sum_i x_i^2\right]\prod_b F(\vec{x}_i,\vec{x}_j)\prod_i d\vec{x}_i\ . \tag{2.4}$$

Henceforth, E denotes $E+i\eta$ and the limit $\eta\to 0^+$ is implied. On a Cayley tree we can integrate each vector $\vec{x}_i$, choosing the polar axis of its neighbor as the z axis. That is, we set $\vec{x}_i\cdot\vec{x}_j=x_ix_j\cos\theta_j$, where θ_j is an angle in the n-component vector space. Then,

$$ig=\frac{1}{n}\int x_0^2\exp\left[\frac{iE}{2}\sum_i x_i^2\right]\prod_i(K_n x_i^{n-1}dx_i)\times\prod_b\langle F(\vec{x}_i,\vec{x}_j)\rangle_{\theta_i}\ , \tag{2.5}$$

where $\langle\ \rangle_{\theta_i}$ denotes an average over θ_j, and $K_n x_i^{n-1}$ is the phase-space element in n dimensions. Consider the bond factor f_{ij} defined by

$$f_{ij}=\langle F(\mathbf{x}_i,\mathbf{x}_j)\rangle_{\theta_i}=\langle[e^{-it_{ij}x_ix_j\cos\theta_j}]_t\rangle_{\theta_j}\ . \tag{2.6}$$

Using the fact that

$$\langle\cos^{2r}\theta\rangle_\theta=\frac{(n/2-1)!(2r)!}{2^{2r}r!(n/2+r-1)!}\ , \tag{2.7}$$

we may express f_{ij} as

$$f_{ij}=f(x_i,x_j)=\sum_{r=0}^{\infty}(-1)^r\left[\frac{x_i^2x_j^2}{4}\right]^r\times\frac{(n/2-1)!}{r!(n/2+r-1)!}[t_{ij}^{2r}]_t\ . \tag{2.8}$$

Now we construct an integral equation from which g can be obtained. For this purpose we first consider a 0-generation tree, by which we mean a tree consisting of a single point for which

$$ig^{(0)=(1/n)}\int x_0^2 e^{(iE/2)x_0^2}d\vec{x}_0=i/E\ , \tag{2.9}$$

where $g^{(m)}$ is the Green's function for an m-generation tree. For a 1-generation tree [see Fig. 2(a)] of coordination number $z=\sigma+1=3$, we have

$$ig^{(1)}=\int x_0^2e^{(iE/2)x_0^2}d\vec{x}_a\left[\int f_{0a}e^{(iE/2)x_a^2}d\vec{x}_a\right]\times\left[\int f_{0b}e^{(iE/2)x_0^2}d\mathbf{x}_b\right]\left[\int f_{0c}e^{(iE/2)x_c^2}d\mathbf{x}_c\right] \tag{2.10a}$$

$$\equiv\frac{1}{n}\int x_0^2e^{(iE/2)x_0^2}[\phi^{(1)}(x_0)]^3d\vec{x}_0\ . \tag{2.10b}$$

More generally, we see that for an m-generation tree with arbitrary coordination number, one has

$$ig^{(m)}=\frac{1}{n}\int x_0^2e^{(iE/2)x_0^2}[\phi^{(m)}(x_0)]^{\sigma+1}d\vec{x}_0\ , \tag{2.11}$$

where $\phi^{(m)}(x)$ satisfies the recursion relation

$$\phi^{(m+1)}(x)=\int e^{(iE/2)y^2}[\phi^{(m)}(y)]^\sigma f(x,y)d\vec{y}\ , \tag{2.12a}$$

with the initial condition

$$\phi^{(0)}(x)=1\ . \tag{2.12b}$$

We now consider the $n\to 0$ limit of Eq. (2.12). For this purpose we write Eq. (2.8) for small n as

$$f(x,y)=1+\frac{2}{n}\sum_{r=1}^{\infty}(-1)^r\left[\frac{x^2y^2}{4}\right]^r[t^{2r}]_t\frac{1}{r!(r-1)!} \tag{2.13a}$$

$$\equiv 1+\frac{1}{n}H(x,y)\ . \tag{2.13b}$$

We insert this form into Eq. (2.12a) and set $d\vec{y}=ny^{n-1}dy$. The integral involving the first term of Eq. (2.13b) is integrated by parts. When the limit $n\to 0$ is taken, we write Eq. (2.12a) as

$$\phi^{(m+1)}(x)=[\phi^{(m)}(0)]^{\sigma}+\int_0^{\infty} e^{(iE/2)y^2}[\phi^{(m)}(y)]^{\sigma} \times H(x,y)\frac{dy}{y} . \quad (2.14)$$

Use of this relation for $x=0$, together with Eqs. (2.12b) and (2.13b), yields $\phi^{(m)}(0)=1$. Now we set

$$x^2=2is/E , \quad y^2=2iu/E , \quad (2.15)$$

so that $\phi^{(m)}(s)$ is determined recursively by

$$\phi^{(m+1)}(s)=1+\int_0^{\infty} e^{-u}[\phi^{(m)}(u)]^{\sigma} \times \sum_{r=1}^{\infty}\frac{u^{r-1}s^r[t^{2r}]_t}{r!(r-1)!E^{2r}}du , \quad (2.16a)$$

with

$$\phi^{(0)}(s)=1 . \quad (2.16b)$$

From the transformation (2.15) and the fact that the imaginary part of E is positive, the path of u integration must end in the fourth quadrant of the complex u plane. A similar transformation shows that the Green's function $g^{(m)}$ for the m-generation tree is

$$g^{(m)}=(1/E)\int_0^{\infty} e^{-u}[\phi^{(m)}(u)]^{\sigma+1}du . \quad (2.17)$$

Thus Eqs. (2.16) and (2.17) have the full information for the Green's function of the random hopping model on the finite Cayley tree.

For an infinite Cayley tree the information is fully contained in the following pair of equations:

$$g=(1/E)\int_0^{\infty} e^{-u}[\phi(u)]^{\sigma+1}du , \quad (2.18)$$

$$\phi(s)=1+\int_0^{\infty} e^{-u}[\phi(u)]^{\sigma}\sum_{r=1}^{\infty}\frac{u^{r-1}s^r[t^{2r}]_t}{r!(r-1)!E^{2r}}du . \quad (2.19)$$

Normally, solution of a nonlinear integral equation such as Eq. (2.19) by iteration is not guaranteed to be valid. Here, however, in view of the interpretation of $\phi^{(m)}(x)$ as being the function for an m-generation tree, we know that this method of solution is correct. Since it seems plausible that $\lim_{m\to\infty}\phi^{(m)}(x)$ exists, we use Eq. (2.19) even though we cannot pretend to have rigorously justified its use.

B. Exact solution for pure case and binary-distribution case

As can be seen from Eq. (2.19), the integral equation is dependent upon $[t^{2r}]_t$ and E^{2r}. Thus the pure case, where the probability distribution is $p(t_{ij})=\delta(t_{ij}-1)$, and the binary-distribution case, where the probability distribution is

$$p(t_{ij})=\tfrac{1}{2}[\delta(t_{ij}-1)+\delta(t_{ij}+1)] ,$$

both have exactly the same integral equation; it is a general property of the random hopping model on two colorable lattices that both the Green's function and the density of states are even functions of energy E. Thus, in both cases, the integral equation is

$$\phi(s)=1+\int_0^{\infty} e^{-u}[\phi(u)]^{\sigma}\sum_{r=1}^{\infty}\frac{u^{r-1}s^r}{r!(r-1)!E^{2r}}du . \quad (2.20)$$

The solution of Eq. (2.20) takes the form

$$\phi(s)=\exp(as/\sigma) . \quad (2.21)$$

If this ansatz for $\phi(s)$ is substituted into Eq. (2.20), then we have

$$\exp(as/\sigma)=1+\int \exp[-u(1-a)]\sum\frac{u^{r-1}s^r}{r!(r-1)!E^{2r}}du , \quad (2.22)$$

which is satisfied, provided that the constant a satisfies the self-consistent equation

$$a=\sigma/E^2(1-a) , \quad (2.23)$$

which is very similar to the mean-field equation for Q in Ref. 16. The solutions of Eq. (2.23) are

$$a_{\pm}=\frac{1}{2}\left[1\pm\left[1-\frac{4\sigma}{E^2}\right]^{-1/2}\right] . \quad (2.24)$$

Of the two choices of sign, a_+ is not a proper solution for the integral equation because it leads to a Green's function for which $Eg(E)$ is not bounded as $E\to\infty$. For $E^2>4\sigma$, the Green's function is real and the density of states vanishes. For $E^2<4\sigma$, the density of states is given by the well-known result,[16] i.e.,

$$\rho(E)=-\frac{1}{\pi}\,\mathrm{Im}g \quad (2.25a)$$

$$=\frac{2\sigma}{\pi}\frac{(\sigma+1)(E_c^2-E^2)^{1/2}}{(\sigma-1)^2E^2+(\sigma+1)^2(E_c^2-E^2)} , \quad E^2<4\sigma \quad (2.25b)$$

$$=0 , \quad E^2>4\sigma \quad (2.25c)$$

where $E_c^2=4\sigma$. The two interesting limiting cases of the formulas (2.25) are (a) the limit $\sigma\to\infty$, and (b) the one-dimensional case which is a special Cayley tree. In the limit $\sigma\to\infty$, $\rho(E)$ obeys

$$\rho(E)=\frac{2}{\pi E_c^2}(E_c^2-E^2)^{1/2} , \quad E^2<E_c^2 \quad (2.26)$$

which is the mean-field solution of Ref. 16, and, as we shall see in the next sections, is the starting point of our corrections. If we define an order-parameter exponent β via

$$\rho(E)\sim|E-|E_c||^{\beta} , \quad |E|\to|E|_c^- \quad (2.27)$$

then β assumes the value $\frac{1}{2}$ for the Cayley tree. As expected, this coincides with the mean-field value of β. In one dimension, Eqs. (2.25) yield the well-known result

$$\rho(E)=(1/\pi)(E_c^2-E^2)^{-1/2} , \quad E^2<E_c^2 . \quad (2.28)$$

C. $1/\sigma$ correction and sum rule

In this subsection we consider perturbative corrections to the density of states away from the $\sigma\to\infty$ solution of Eq. (2.26). We set $E^2=\sigma e^2$ in the integral equation (2.19) and look for a solution to the integral equation in the form

$$\phi(s)=\exp\left[\sum_{r=1}^{\infty} a_r(s/\sigma)^r\right], \tag{2.29}$$

where

$$a_k=a_k^{(0)}+a_k^{(1)}/\sigma+a_k^{(2)}/\sigma^2+\cdots . \tag{2.30}$$

If the form of Eq. (2.29) is substituted into the integral equation, we obtain

$$\phi(s)=1+\int_0^{\infty}\exp\left[-u+a_1u+a_2\frac{u^2}{\sigma}+\cdots\right]\times\sum_{r=1}^{\infty}\frac{u^{r-1}s^r[t^{2r}]_t}{(r-1)!r!(\sigma e^2)^r}du . \tag{2.31}$$

In view of Eq. (2.29), this is equivalent to a set of nonlinear equations for the coefficients a_r, which we solve perturbatively in the parameter $1/\sigma$. Initially, we consider the asymptotic behavior for $\sigma\to\infty$. To lowest order in $1/\sigma$, we drop all correction terms in the expansion of the argument of the exponential function in Eq. (2.31), which then becomes $\exp[(a_1-1)u]$. Then, from this equation we find that a_1 satisfies the self-consistent relation

$$a_1=[t^2]_t/e^2(1-a_1) . \tag{2.32}$$

In view of Eq. (2.29), we determine the a_k's by determining $\phi(s)$. This is done by evaluating Eq. (2.31) to lowest order in $1/\sigma$,

$$\phi(s)=1+\sum_{r=1}^{\infty}\left[\frac{s}{\sigma e^2(1-a_1)}\right]^r\frac{[t^{2r}]_t}{r!} . \tag{2.33}$$

Since a_1 is known from Eq. (2.32), this equation is, in essence, an evaluation of the a_k's. Equation (2.32) is analogous to Eq. (2.23), so that within the zeroth-order approximation of the $1/\sigma$ expansion the density of states has the same semicircular form as that in Eq. (2.26), except that the critical energy E_c, in general, is given by

$$E_c^2=4\sigma[t^2]_t . \tag{2.34}$$

From Eq. (2.32) it follows that the critical exponent for the order parameter is $\beta=\frac{1}{2}$.

For the correction, proportional to $1/\sigma$, to mean-field theory, we should include corrections to Eq. (2.32) linear in a_2. Then the self-consistent equations for a_1 and a_2 obtained by analyzing Eq. (2.31) to order s and s^2, respectively, are

$$a_1=\frac{[t^2]_t}{e^2(1-a_1)}+\frac{1}{\sigma}\frac{2[t^2]_t a_2}{e^2(1-a_1)^3} , \tag{2.35a}$$

$$a_2+\tfrac{1}{2}a_1^2=\frac{[t^4]_t}{2e^4(1-a_1)^2}+\frac{1}{\sigma}f(a_1,a_2) , \tag{2.35b}$$

where the explicit form of f will not be needed. It is important to note that for $\sigma\gg1$ we can always use Eq. (2.35b) to eliminate a_2 from Eq. (2.35a). In this way we generate corrections of order $1/\sigma$ to Eq. (2.32) as

$$a_1=\frac{[t^2]_t}{e^2(1-a_1)}+\frac{1}{\sigma}\frac{\delta_t a_1^2}{e^2(1-a_1)^3} , \tag{2.36}$$

where

$$\delta_t=\frac{[t^4]_t-([t^2]_t)^2}{[t^2]_t} . \tag{2.37}$$

It is now simply a straightforward calculation to obtain corrections to a_1, E_c, and $\rho(E)$ to order $1/\sigma$. This is done in Appendix A, where, for instance, we find

$$\rho(E)=\begin{cases}(2/\pi E_c^2)(E_c^2-E^2)^{1/2}+(1/\sigma)\rho^{(1)}(E) , & E^2<E_c^2\\ 0 , & E^2>E_c^2\end{cases} \tag{2.38}$$

where

$$\rho^{(1)}(E)=\frac{2}{\pi E_c}\left[1-\frac{\delta_t}{[t^2]_t}\right]\left[\frac{E_c^2-E^2}{E_c^2}\right]^{1/2}\left[4\frac{E^2}{E_c^2}-1\right] \tag{2.39}$$

and

$$E_c^2=4\sigma[t^2]_t\left[1+\frac{1}{\sigma}\frac{\delta_t}{[t^2]_t}\right] . \tag{2.40}$$

The density of states (2.38) satisfies the sum rule $\int\rho(E)dE=1$ exactly, because

$$\int\rho^{(1)}(E)dE=0 . \tag{2.41}$$

More generally, as in ordinary perturbation theory, one can show that all corrections proportional to $(1/\sigma)^n$ satisfy Eq. (2.41), and, hence, that a perturbative calculation preserves the sum rule on $\rho(E)$.

From the form of Eq. (2.38) we see that the corrections perturbative in $1/\sigma$ are irrelevant (in the sense of critical phenomena). They lead to (a) a perturbative shift in E_c, so that the exact values of E_c, the density of states, etc. depend on all the moments $[t^{2r}]_t$; (b) a modification in the amplitude of the singularity $\sim(E_c^2-E^2)^{1/2}$; and (c) corrections to the dominant singularity which are of order $(E_c^2-E^2)^{(1+m)/2}$ with $m=1,2,\ldots$; but (d) no renormalization of the mean-field exponent $\beta=\frac{1}{2}$ which describes the sharp transition in $\rho(E)$, shifted as described in (a)–(c) above.

III. NONPERTURBATIVE CORRECTIONS FOR THE GAUSSIAN DISTRIBUTION

A. Shift of the critical energy to the unphysical sheet

In this section we will show that the critical energy E_1 at which the Green's function is singular shifts to the unphysical sheet of energy where the imaginary part of E is

negative. From this fact one sees why there is no sharp transition on the real energy axis, in accord with Wegner's theorem,[19] as was stated in the Introduction. To do this, we study the nonperturbative corrections. For this analysis it is convenient to restrict our attention to the Gaussian distribution for the random hopping constant:

$$p(t_{ij})=(1/2\pi)^{1/2}\exp(-t_{ij}^2/2)\ , \tag{3.1}$$

where we set the dispersion of the distribution equal to 1. In this case, $[t^{2r}]_t$ is

$$[t^{2r}]_t=2r!/2^r r!\ , \tag{3.2}$$

and it increases faster than geometrically as r increases. Inserting this evaluation into Eqs. (2.32) and (2.33) gives, to lowest order in $1/\sigma$,

$$\phi(s)=(1-2a_1 s/\sigma)^{-1/2}\ ,\quad 2a_1 s/\sigma<1\ . \tag{3.3}$$

Up to now we have treated Eq. (2.31) perturbatively. In such a calculation it is obvious that to all orders in $1/\sigma$ all the a_k's will remain real at large energy, and therefore the density of states will be zero term by term in $1/\sigma$. A nonzero imaginary part to a_k must come from a saddle-point contribution to the integral in Eq. (2.31). Qualitatively speaking, the saddle-point contributions can be obtained using the lowest-order approximation, Eq. (3.3), for $\phi(s)$. We will outline the calculation of these effects when $\phi(s)$ is treated perturbatively. In essence, this amounts to treating the interaction between perturbative and saddle-point effects.

We now turn to an analysis of Eq. (2.31) which includes both perturbative effects discussed in the preceding section and also the saddle-point contributions which describe the appearance of eigenstates at arbitrarily large energy for the Gaussian random problem. The integration in Eq. (2.31) should be carried over the steepest-descent path in the complex s plane, which passes through the stationary phase point of the integrand at $s=s^*$, as shown in Fig. 3. Here the calculations are for the outside of the band, i.e., the range of energy where all a_k's are real if the nonperturbative corrections are not included. The contour of integration should be deformed from the real axis in order to avoid the divergence at $s=\sigma/(2a_1)$ and must end in the fourth quadrant, as was stated in Sec. II A. Thus, using Eqs. (2.31) and (3.3), we find that the coefficients a_k obey the following relation:

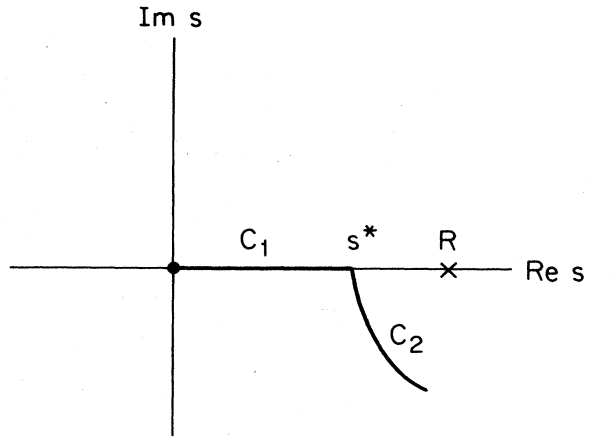

FIG. 3. The contour C_1+C_2 for the integrations of Eqs. (4.76) and (4.77) in the complex s plane. The saddle point is at s^* and the point $s=R=\sigma/(2a_1)$ is where $\phi(s)$ in Eq. (4.75) is singular.

$$a_1=\frac{1}{e^2}\int_{C_1}\exp[-s(1-a_1)]\left[1+\frac{a_2 s^2}{\sigma}+\cdots\right]ds + \frac{1}{e^2}\int_{C_2}\exp\left[-s+\sum_n a_n(s/\sigma)^n\right]ds\ , \tag{3.4a}$$

$$a_2+\tfrac{1}{2}a_1^2=\frac{3}{2e^4}\int_{C_1}\exp[-s(1-a_1)]\times\left[1+\frac{a_2 s^2}{\sigma}+\cdots\right]s\,ds + \frac{3}{2e^4}\int_{C_2}\exp\left[-s+\sum_n a_n(s/\sigma)^n\right]s\,ds\ , \tag{3.4b}$$

etc. Here we have explicitly separated the integral into the perturbative part (along the contour C_1), which gives rise to Eq. (2.35), and the saddle-point contribution (from the contour C_2). To discuss Eqs. (3.4), we introduce the notation

$$I(C_1;f(s))=\int_{C_1}\exp[-s(1-a_1)]f(s)ds\ , \tag{3.5a}$$

$$I(C_2;f(s))=\int_{C_2}\exp\left[-s+\sum_n a_n(s/\sigma)^n\right]f(s)ds\ . \tag{3.5b}$$

Equations (3.4) are then

$$F_1(\{a\})=a_1-e^{-2}I(C_1;1+a_2 s^2/\sigma+a_3 s^3/\sigma^2+\cdots)-e^{-2}I(C_2;1)=0\ , \tag{3.6a}$$

$$F_2(\{a\})=a_2-\tfrac{1}{2}a_1^2-\tfrac{3}{2}e^{-2}I(C_1;s+a_2 s^3/\sigma+a_4 s^3/\sigma^2+\cdots)-\tfrac{3}{2}e^{-2}I(C_2;s)=0\ , \tag{3.6b}$$

$$F_3(\{a\})=a_3-a_1 a_2-\tfrac{1}{6}a_1^3-\tfrac{5}{4}e^{-2}I(C_1;s^2+a_2 s^4/\sigma+a_3 s^5/\sigma^2+\cdots)-\tfrac{5}{4}e^{-2}I(C_2;s^2)=0\ . \tag{3.6c}$$

These equations can be analyzed perturbatively in $1/\sigma$ by using the equations for F_m to eliminate a_m in favor of a_k with $k<m$, for all $m>1$. We can do this without encountering any singularity because all the Jacobians

$$\frac{\partial(F_2,F_3,\ldots,F_m)}{\partial(a_2,a_3,\ldots,a_m)} \tag{3.7}$$

differ from unity by terms of order $1/\sigma$. Thus we can focus only on the equation for a_1 and imagine that higher a_k's have been expressed in terms of a_1. We work to first order in $1/\sigma$, and by using Eq. (3.4b) to determine a_2, we obtain, from Eq. (3.4a),

$$a_1-\frac{1}{e^2(1-a_1)}+\frac{2a_1^2}{\sigma e^2(1-a_1)^3}+\frac{1}{e^2}\int_{C_2}\exp[-\psi(s)]ds + \frac{1}{e^4(1-a_1)^2}\left[\left[1-\frac{2s^*}{\sigma e^2(1-a_1)}\right]^{-3/2}-1\right]\int_{C_2}\exp[-\psi(s)]ds \equiv F(a_1,e)=0 , \tag{3.8}$$

where the first two terms are essentially the same as Eq. (2.36) with $[t^2]_t=1$ and $[t^4]_t=3$, the third term is the direct contribution to a_1 from the integral along C_2, and the last term is the contribution from a_2 involving the integral along C_2. Here, $\psi(s)$ is defined as

$$\psi(s)\equiv s-\sigma\ln\phi(s) . \tag{3.9}$$

Using Eq. (3.3) as the leading-order approximation, the saddle point s^* is determined by

$$\psi'(s^*)=1-\tfrac{1}{2}\sigma\frac{2a_1/\sigma}{1-2a_1s^*/\sigma}=0 , \tag{3.10}$$

and thus

$$s^*=\sigma/2a_1(1-a_1) . \tag{3.11}$$

If one considers Eq. (3.8) as the equation of state for a_1, then the critical values a_{1c} and e_1 (i.e., $E_1/\sigma^{1/2}$) are determined by

$$F(a_{1c},e_1)=0 , \quad \frac{\partial F(a_{1c},e_1)}{\partial a_1}=0 . \tag{3.12}$$

From Eq. (2.32), we know that in the limit $\sigma\to\infty$, $e_c^\infty=2$ and $a_{1c}^\infty=\frac{1}{2}$. To solve Eq. (3.8), we set

$$F(a_1,e)=F_0(a_1,e)+\delta F(a_1,e) ,$$

where

$$F_0(a_1,e)=a_1-[e^2(1-a_1)]^{-1} .$$

Then it is easily shown that, for $\delta F\ll 1$, e_1 is given, from Eq. (3.12), by

$$e_1=e_c^\infty-\delta F/(\partial F_0/\partial e)\,|_{a_1=a_{1c}^\infty;\,e=e_c^\infty} . \tag{3.13}$$

From Eq. (3.8) we have

$$\delta F=-\frac{1}{2\sigma}-2^{-3/2}\int_{C_2}e^{\psi(s)}ds . \tag{3.14}$$

Thus Eq. (3.13) yields

$$e_1=2(1+1/\sigma)-i\gamma . \tag{3.15}$$

Here, $i\gamma$ comes from the second term in Eq. (3.14), which for large σ is asymptotically

$$\gamma=(\pi\sigma/2)^{1/2}2^{-\sigma/2}\exp(-\sigma/2) . \tag{3.16}$$

Recall that we are studying $G(0,0;E+i\eta)$, which must be analytic for $\mathrm{Im}E>0$. Then Eq. (3.15) indicates that the critical energy E_1 is shifted to the unphysical sheet ($\gamma>0$). This means that physical E can never reach E_1, at which the Green's function is singular. This shows why there is no sharp transition on the real axis of the energy E, and the density of states never vanishes at real energy E, in accord with Wegner's theorem.[19]

B. Density of states for $E^2>E_c^2$

Since the critical energy E_1 has a nonvanishing imaginary part, the density of states for $E^2>E_c^2=(\mathrm{Re}E_1)^2$ does not vanish. In this subsection we briefly describe the behavior of the density of states, $\rho_>(E)$, for $E^2>E_c^2$. Detailed calculations for this subsection are given in Appendix B.

For $E^2\approx E_c^2$, the density of states is

$$\rho_>(|E|\sim|E_c|)\approx cE^{-1/2}\exp\left[-\frac{E^2}{16}\right] \approx c\left[\frac{\exp(-\sigma)}{\sigma}\right]^{1/4} , \tag{3.17}$$

where c is a constant. In this limit, $\rho(E)$ is dominated by the coupling between the perturbative correction and the nonperturbative correction. For $E^2\gg E_c^2$, the density of states approaches

$$\rho_>(E)=\left[\frac{\exp(1)}{4\pi}\right]^{1/2} \times\left[\frac{E^2}{\sigma}\right]^{\sigma/2}\exp(-E^2/2)(1+\mathrm{const}\times E^{-2}) , \tag{3.18}$$

which comes dominantly from the high-energy nonperturbative correction to the integral for the Green's function $g(E)$. Thus there is a crossover from the high-energy regime in which only the nonperturbative correction is dominant to the "critical regime" in which the coupling between the nonperturbative correction and the perturbative correction is dominant. The energy δE around E_c which defines this crossover regime is given by (see Fig. 4)

$$E^2-E_c^2\sim E_c\delta E\sim\tfrac{1}{2}(\pi\sigma^3)^{1/2}2^{\sigma/2}\exp(-\sigma/2) . \tag{3.19}$$

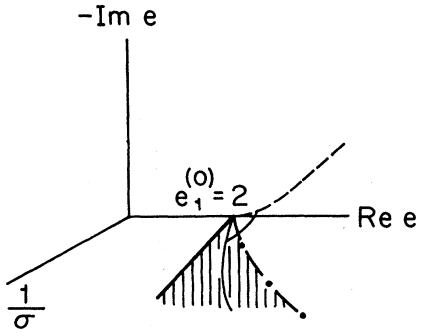

FIG. 4. Shift of critical energy E_1 to unphysical sheet and the crossover indicated in Eq. (3.19). Here, $e_1^{(0)}=2$ is the mean-field value of the critical energy. (1) The solid line on the $\mathrm{Re}(e)-1/\sigma$ plane describes the perturbative $1/\sigma$ correction, i.e., $e_c=2(1+1/\sigma)$. (2) The dashed curve for which $\mathrm{Im}e\neq 0$ denotes the nonperturbative shift to the unphysical sheet. (3) The lighter curved line represents the real part of e_1 including a nonperturbative shift which we have not calculated. The hatched part of the $\mathrm{Re}(e)-1/\sigma$ plane (the region between the solid line and the dashed-dotted curve) is the domain in which the coupling between the perturbative method and the nonperturbative method is dominant. The other part of the plane, $e>e_c$, is where the nonperturbative method is dominant.

Equation (3.18) is exactly the same result as that derived by a variational argument in Ref. 16. We also conclude that formula (3.18) for large energy E holds for the hypercubic lattices as well, because when a hopping constant t_{ij} is large and comparable to E, it should be surrounded by bonds with smaller t_{ij}'s, and thus there should be no possibility of the loop corrections.

If the next-order corrections are included, then the density of states $\rho_>(E)$ is

$$\rho_>(E)=\rho_>(\sigma e^2)\sim\exp[-c\sigma(1+c'\sigma^{-1}+\cdots)] . \quad (3.20)$$

Therefore, in the limit $\sigma\to\infty$, we need not consider them. In the region $E^2<E_c^2$, similar nonperturbative corrections to the result (2.38) must exist, but we know that from the sum rule that this correction should be order of $\exp(-c\sigma)$, and the effect from this correction should hardly be distinguishable from the result (2.38) in this range. Summarizing all the results for the Gaussian distribution case, the density of states for $E^2<E_c^2$ is

$$\rho_<(E)=\frac{2}{\pi E_c}\left[\frac{E_c^2-E^2}{E_c^2}\right]^{1/2}\times\left[1+\frac{1}{\sigma}\frac{E_c^2-4E^2}{E_c^2}\right]+O(1/\sigma^2)+\rho_<^{\mathrm{non}}(E) , \quad (3.21)$$

where $\rho_<^{\mathrm{non}}$ is the nonperturbative correction for $E^2<E_c^2$, which is the order of $\exp(-c\sigma)$, and $\rho_<^{\mathrm{non}}(E_c)=\rho_>(E_c)$. Thus, in the limit $\sigma\to\infty$, the density of states is a smooth function of E at $E=E_c$. This means that as far as the density of states is concerned, there is no sharp transition on the real energy axis of E, in accord with Wegner's theorem.[19] The shift of E_1 to the unphysical sheet of E provides a mechanism for this result within the language of critical phenomena.

IV. BOUNDED-DISTRIBUTION CASE

A. Bound-state analysis

By a bounded distribution we mean one in which the probability distribution of the random hopping constants $p(t_{ij})$ is nonzero only for some bounded range of t_{ij}. One typical example in this category is the rectangular distribution

$$p(t_{ij})=\begin{cases}\frac{1}{2} & \text{for } -1<t_{ij}<1 ,\\ 0 & \text{otherwise} ,\end{cases} \quad (4.1)$$

which was first used in Anderson's historic paper.[1] The moments $[t^{2r}]_t$ for this distribution are given by

$$[t^{2r}]_t=1/(2r+1) , \quad (4.2)$$

and they decrease as r increases. Because of this fact, the zeroth-order approximation for $\phi(s)$ is a power series which is convergent for $-\infty<s<\infty$. From Eq. (2.33) we obtain

$$\phi(s)=1+\sum_r\frac{1}{r!(2r+1)}\left[\frac{s}{\sigma e^2(1-a_1)}\right]^r . \quad (4.3)$$

Thus the coefficients a_k $(k\geq 4)$ in this approximation are negative. Thus, quite contrary to the Gaussian distribution case, the integral (2.19) for the Green's function g is some positive definite real number in the range $E^2>E_c^2$, and this seems true for any correction proportional to $(1/\sigma)^n$, where n is finite. Thus, from Eqs. (2.40) and (4.2), the density of states of this finite-distribution case seems to be nonzero only for $|E|<E_c$, where it is given by

$$\rho(E)=\frac{2}{\pi E_c^2}(E_c^2-E^2)^{1/2}\left[1+\frac{1}{5\sigma}\left[4\frac{E^2}{E_c^2}-1\right]\right] , \quad (4.4)$$

with

$$E_c^2=(4\sigma/3)(1+4/5\sigma) , \quad (4.5)$$

where we have included corrections proportional to $1/\sigma$. At this level of approximation it seems that for the Cayley tree there is a sharp transition on the real axis of E at a critical value of energy E_c given by Eq. (4.5). If true, this result would indicate that $E_c^2<4\sigma$, where 4σ is the exact critical energy if all the hopping constants t_{ij} are equal to 1 (i.e., for the nonrandom case). However, as we shall see, following the reasoning used for the three-dimensional lattice by Lifschitz,[26] this is not the case: There are states at all the energies up to $E^2=4\sigma$.

The argument of Lifschitz is essentially the following.[26] There is a finite, albeit very small, probability of having a sphere of radius R within which all the interactions are at least as large as $t_>$ for any $t_>$ <1. For R large and $t_>$ near 1, this region has states arbitrarily close to the exact band edge, and this argument gives the asymptotic form of the density of states. We wish to formulate this argu-

ment for the Cayley tree. For the Cayley tree the surface area is anomalously large compared to the volume, and thus the status of the Lifschitz argument is not *a priori* obvious. To apply the Lifschitz argument to the Cayley tree, we should analyze the situation as in Fig. 5, where the bonds from the center site to the mth generation have the same hopping constant $t_>$, and the remaining bonds have a typical hopping constant, which we denote by $t_<$ in order to emphasize that it is definitely less than $t_>$. This shape region, within which the coupling constant is anomalously large, is very similar to the spherical volume used by Lifschitz, inside which there are only the same impurity atoms. To minimize the anomalous effects of the large surface-to-volume ratio for the Cayley tree, we consider the effect of this region on the local Green's function g for the site at the center of the region. In essence, then, our calculation will only be sensitive to totally symmetric states. To study such bound states for this special situation, we should modify Eqs. (2.16) and (2.19).

Let us start from simple situations like Figs. 6(a) and 6(b), and, for simplicity, we consider initially the case $\sigma=2$. The function $\phi(s)$ for the tree shown in Fig. 6(a) is found, using the method of Sec. II A, to be

$$\phi^{1,0}(s)=1+\int_0^\infty e^{-u}(1)^2\sum_{r=1}^{\infty}\frac{s^r u^{r-1}t_>^{2r}}{r!(r-1)!E^{2r}}du\ , \tag{4.6}$$

where the first superscript gives the number of generations starting from the origin which are occupied by the hopping constant $t_>$, and the second superscript gives the number of outer generations occupied by $t_<$. The function $\phi(s)$ for the tree shown in Fig. 6(b) is

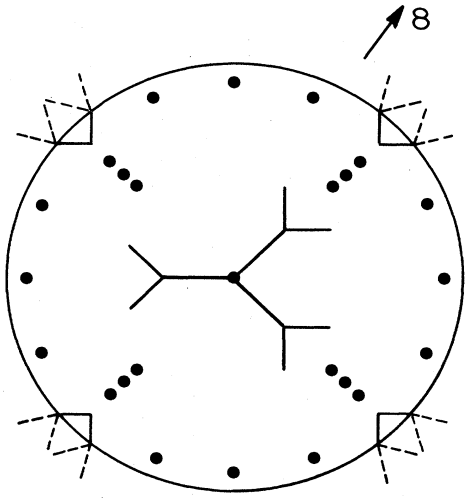

FIG. 5. Spherical configuration for Lifschitz argument on the Cayley tree. Inside the circle there are m generations of bonds, all of whose hopping constants are $t_>$. (Bonds with $t_>$ are represented by a solid line.) The hopping constants for the remaining bonds in the infinite number of generations outside the circle are $t_<$. (Bonds with $t_<$ are represented by dashed lines.)

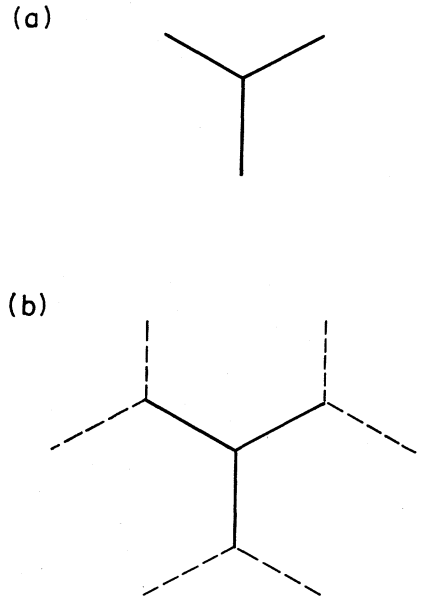

FIG. 6. (a) 1-generation tree with only $t_>$ (solid line). (b) 2-generation tree with $t_>$ (solid line) for inner bonds and $t_<$ (dashed line) for outer bonds.

$$\phi^{1,1}(s)=1+\int_0^\infty e^{-u}[f(u)]^2\sum_{r}\frac{s^r u^{r-1}t_>^{2r}}{r!(r-1)!E^{2r}}du\ , \tag{4.7}$$

where

$$f(u)=1+\int_0^\infty e^{-p}(1)^2\sum_{r=1}^{\infty}\frac{u^r p^{r-1}t_<^{2r}}{r!(r-1)!E^{2r}}dp\ . \tag{4.8}$$

We generalize the recursion relation for $\phi^{m,n}$ from the simple results of Eqs. (4.6) and (4.7). It is convenient to define the following nonlinear integral operators I and J:

$$If(s)\equiv 1+\int_0^\infty e^{-u}[f(u)]^\sigma\sum_{r=1}^{\infty}\frac{s^r u^{r-1}t_>^{2r}}{r!(r-1)!E^{2r}}du \tag{4.9}$$

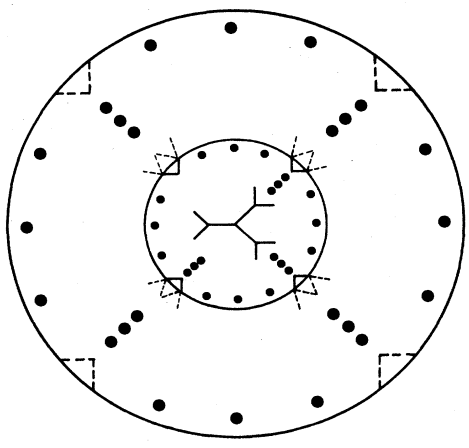

FIG. 7. m,n-generation tree in which the bonds of the m generations inside the inner circle have $t_>$ (solid line) and the bonds of the outer n generations have $t_<$ (dashed line).

and

$$Jf(s)\equiv 1+\int_0^\infty e^{-u}[f(u)]^\sigma \sum_{r=1}^{\infty}\frac{s^r u^{r-1} t_<^{2r}}{r!(r-1)!E^{2r}}du \ . \quad (4.10)$$

Then the function $\phi^{m,n}(s)$ for the general situation like Fig. 7 where the inner m generations are occupied by $t_>$ and the outer n generations are occupied by $t_<$ (hereafter, we call this the m,n-generation tree) is

$$\phi^{m,n}(s)=I^m[J^n\phi^{0,0}(s)] \ , \quad (4.11)$$

where $\phi^{0,0}(s)=1$. Thus the $\phi^{m,\infty}(s)$ for the Lifschitz situation like Fig. 5 is

$$\phi^{m,\infty}(s)=I^m[J^\infty(1)] \ . \quad (4.12)$$

From the form of nonlinear integral operators I and J, we know that the form of the function $\phi^{m,n}(s)$ for the m,n-generation tree is

$$\phi^{m,n}(s)=\exp(a_{m,n}s/\sigma) \ , \quad (4.13a)$$

with

$$a_{0,0}=0 \ . \quad (4.13b)$$

Then the recursion relation for the coefficient $a_{m,n}$ is found from Eqs. (4.10), (4.11), and (4.1) to be

$$a_{m,n}=\frac{t_>^2}{e^2(1-a_{m-1,n})} \ , \quad (4.14a)$$

$$a_{0,n}=\frac{t_<^2}{e^2(1-a_{0,n-1})} \ , \quad (4.14b)$$

where E^2 has been replaced by σe^2 as before. Thus, from Eqs. (4.13) and (2.17), the Green's function of the m,n-generation tree $g^{m,n}$ is

$$Eg^{m,n}=\frac{1}{1-\frac{\sigma+1}{\sigma}a_{m,n}} \ . \quad (4.15)$$

Thus the simple pole condition of Eq. (4.15) satisfies

$$1-\frac{\sigma+1}{\sigma}a_{m,n}(e)=0 \ , \quad (4.16)$$

and the solutions of this equation, $e_1,e_2,\ldots$, are the energies of the "s-wave" bound states for the m,n-generation tree. (By an "s-wave" state we mean one which is constant within each generation.) Thus, if one solves Eqs. (4.14) and (4.16), then the distribution in energy of the bound states at the origin of the m,n-generation tree can be analyzed.

B. Lifschitz tail

In this subsection, using the analysis of the preceding subsection and the Lifschitz argument, we show that the density of states never vanishes within the band, and, in addition, we analyze how the density of states behaves at the band edge. Here, by within the band we mean in the energy range $E^2<4\sigma t_{\max}$, where $t_{\max}$ is the maximum nonvanishing hopping constant in the finite distribution. For the rectangular distribution (4.1), $t_{\max}=1$. Since we are interested in the behavior at the band edge, we can assume that $t_>>t_<$ and $e^2>4t_<^2$, so that e is outside the band, corresponding to a pure system with hopping constant $t=t_<$. For convenience, we will assume that $t_>=1$. Thus we treat the case when

$$4t_<^2<e^2<4t_>^2=4 \ , \quad (4.17a)$$

$$t_<<t_>=1 \ . \quad (4.17b)$$

We start our analysis by writing $\phi^{m,\infty}(s)$ for the m,∞-generation tree as

$$\phi^{m,\infty}(s)=\exp(a_{m,\infty}s/\sigma) \ , \quad (4.18)$$

where $a_{m,\infty}$ satisfies the following recursion relation:

$$a_{m,\infty}=\frac{1}{e^2(1-a_{m-1,\infty})} \ . \quad (4.19a)$$

If we set $a_{m,\infty}\equiv B_m/e$, then we can write B_m as the following continued fraction:

$$B_m=\cfrac{1}{e-\cfrac{1}{e-\cfrac{1}{e-\cfrac{1}{\ddots\cfrac{}{e-ea_{0,\infty}}}}}} \quad (4.19b)$$

where $a_{0,\infty}$ is the stable fixed-point value of the second recursion relation in Eq. (4.14). Thus,

$$B_0=ea_{0,\infty}=\frac{e-(e^2-4t_<^2)^{1/2}}{2} \ , \quad (4.20)$$

which is real because we are interested in the region $e^2>4t_<^2$. If $t_<=0$, Eq. (4.20) reduces, as it should, to the result for an m,0-generation tree, $a_{0,\infty}=B_0=0$. As we shall see, the initial values of the recursion relation (4.18), $a_{0,\infty}$, are physically unimportant for our analysis. Using the general treatment of continued fractions in Ref. 30, we can write $B_m(e)$ as

$$B_m(e)=\frac{C_m(e)-B_0C_{m-1}(e)}{D_m(e)-B_0D_{m-1}(e)} \ , \quad (4.21)$$

where C_m and D_m satisfy the difference equations ($m\geq 1$)

$$C_{m+1}-eC_m+C_{m-1}=0 \ , \quad D_{m+1}-eD_m+D_{m-1}=0 \quad (4.22)$$

with the initial conditions $C_1=1$, $C_0=0$, $D_1=e$, and $D_0=1$. Solving this difference equation, we obtain

$$B_m=\frac{x_1^m-x_2^m-B_0(x_1^{m-1}-x_2^{m-1})}{(x_1^{m+1}-x_2^{m+1})-B_0(x_1^m-x_2^m)} \ , \quad (4.23)$$

where $x_{1,2}$ are given by

$$x_{1,2}=[e\pm(e^2-4)^{1/2}]/2 \ , \quad (4.24)$$

and are the solutions of the quadratic equation $X^2-eX+1=0$. If $e>2$, then $x_2/x_1<1$, and thus $B_m<e/2$. Since the pole condition (4.16) cannot be satisfied in this case, we conclude that there are no states for

$e>2$ (i.e., $E^2>4\sigma$). If $e<2$, x_1 and x_2 can be written as

$$x_{1,2}=\cos\theta\pm i\sin\theta\ , \tag{4.25}$$

where $\cos\theta=e/2$. In this case the pole condition (4.16) becomes

$$a_{m,\infty}=\frac{\sin(m\theta)-B_0\sin[(m-1)\theta]}{2\cos\theta\{\sin[(m+1)\theta]-B_0\sin(m\theta)\}}=\frac{\sigma}{\sigma+1}\ . \tag{4.26}$$

We are interested in the band edge, where $e\approx 2$, which, by Eqs. (4.24) and (4.25), corresponds to small θ. Accordingly, we look for the solution θ to Eq. (4.26) in the limit $m\to\infty$ with $\theta\ll 1$. There is no solution for which $m\theta\to 0$ as $m\to\infty$. The smallest solution for large m satisfies

$$\theta_m=\frac{\pi}{m}-\frac{\text{const}}{m^2}+O\left[\frac{1}{m^3}\right]\ ,\quad \sigma>1 \tag{4.27a}$$

$$\theta_m=\frac{\pi}{2m}-\frac{\text{const}}{m^2}+O\left[\frac{1}{m^3}\right]\ ,\quad \sigma=1\ \text{(one dimension)}\ . \tag{4.27b}$$

Thus the bound-state energy E_m for the m,∞-generation tree nearest the band edge satisfies

$$\delta E=E_{\text{edge}}-E_m=\text{const}/m^2 \tag{4.28}$$

in the limit $m\to\infty$. If the probability is p for a bond to have a hopping constant at least as large as $t_>$, then the probability for the m,∞-generation tree is proportional to $p^{(\sigma^m)}$. From this fact and Eq. (4.28), the density of states near the band edge is

$$\rho(E)\sim\exp(-S\sigma^{c(E_{\text{edge}}-E)^{-1/2}})\ ,\quad E\to E_{\text{edge}} \tag{4.29}$$

where the constant S is dependent upon the boundary shape of the geometrical configuration,[26,31] and in our spherical-shape case it is of order $|\ln p|$. In one dimension where $\sigma=1$, our analysis exactly reproduces the result of Lifschitz, and

$$\rho(E)\sim\exp[-S(E_{\text{edge}}-E)^{-1/2}]\ ,\quad E\to E_{\text{edge}}\ \text{(one dimension)}\ . \tag{4.30}$$

In view of Eq. (4.29), we conclude that throughout the band (i.e., $E^2<4\sigma$), the density of states is nonzero, and, as is the case for the hypercubic lattices, we believe that the formula (4.29) smoothly meets the formula (4.4) near $|E_c|$ given by Eq. (4.5). Therefore there is no sharp transition inside the band, in accord with Wegner's theorem.[19]

C. Dynamical-system interpretation

In this subsection, using the dynamical-system language,[27–29] we explain why there is no density of states outside the band and why the density of states never vanishes inside the band. We consider the recursion relation (4.19a) as a one-dimensional mapping of the form

$$X_{m+1}=f(X_m)=1/e^2(1-X_m)\ , \tag{4.31}$$

where e^2 is the parameter of mappings. In the regime $e^2>4$, the fixed points of the mapping are

$$X^*_{\pm}=\frac{1}{2}\left[1\pm\left[\frac{e^2-4}{e^2}\right]^{1/2}\right]\ , \tag{4.32}$$

where X^*_+ is not stable since $f'(X^*_+)>1$. Thus the unique stable fixed point of the mapping with parameter $e^2>4$ is X^*_-. Thus, whatever the initial values of mapping, i.e., whatever the value of B_0 or $a_{0,\infty}$ may be, the iteration of the mapping will drive X_m, i.e., $a_{m,\infty}$ arbitrarily close to the fixed point X^*_-. However, the pole condition (4.16) is never satisfied at the fixed point X^*_- because $X^*_-<\frac{1}{2}$. Thus there must be no density of states in the region $e^2>4(E^2>4\sigma)$. In the regime $e^2<4$, the fixed points are complex:

$$X^*_c=\frac{1}{2}\left[1-i\left[\frac{4-e^2}{e^2}\right]^{1/2}\right]\ . \tag{4.33}$$

Therefore, if the initial value of X is real, neither of the complex-valued fixed points are accessible. If X is started with an infinitesimal imaginary part, it will evolve under iteration to the fixed point it can reach without crossing the real axis. It is interesting to consider in more detail what happens when the initial value of X is real. Since physical properties should be insensitive to the initial starting point, as we have seen in the preceding subsection, we may take the initial value of the mapping to be $X_0=0$. Then from Eq. (4.26), after m iterations, X_m becomes

$$X_m=\frac{\sin(m\theta)}{2\cos\theta\sin[(m+1)\theta]}\ . \tag{4.34}$$

Thus for large m, X_m is a very rapidly fluctuating function of θ. Thus from the pole condition (4.16) [which in this case is $X_m=\sigma/(\sigma+1)$], the solutions of $e=2\cos\theta$ have a more and more dense distribution in $-2<e<2$ as m becomes larger and larger. Thus, even though the probability of occurrence of the m,∞-generation tree becomes smaller and smaller as m increases, there should be the nonvanishing density of states between $-2<e<2$. That explains why there is no density of states outside the band and no sharp transition inside the band.

V. DISCUSSION AND CONCLUSION

Even though we have not explicitly displayed, from the integral-equation formulation, the shift of E_1 to the unphysical sheet for the bounded distribution, we still believe that it occurs because the density of states is nonzero out to the exact band edge. From Eq. (4.29), the density of states near the band edge can be written as

$$\rho(E)\sim\exp(-S\sigma^{c\sigma^{-1/4}(e_{\text{edge}}-e)^{-1/2}})\ , \tag{5.1}$$

where we set $E^2=\sigma e^2$. To obtain this result from the nonperturbative calculation used in Sec. III, one would have to keep infinitely many terms of $1/\sigma$ expansion of the exponential function in Eq. (2.30). Because we cannot do this effectively, the integral-equation formulation seems to fail to predict the shift for a bounded distribu-

tion. However, we believe that as far as the density of states is concerned, there is no sharp transition inside the band, and also that there is a shift of the critical energy to the unphysical sheet of complex energy.

Another point we want to discuss concerns the mapping (4.31). It has one stable fixed point for a certain range of parameter e and is completely chaotic for real e for the remaining range. There are no bifurcations, no noise cycles, etc.,[27] as found in dynamical systems. The mapping also suggests a very good theoretical mechanism for the band formation, and it is very similar in a sense to the mapping used in Ref. 29 to describe the dynamical behavior of the dilute Heisenberg model on the one-dimensional chain. More study is needed in this context.

In summary, we conclude the following.

(1) The integral-equation formulation for the configurationally averaged one-particle Green's function g of the random hopping model on the Cayley tree is established through the replica method, and the well-known result for the configurationally averaged density of states,

$$\rho(E)\sim(E_c-E)^{1/2}\ ,\quad E\to E_c \tag{5.2}$$

is obtained in the limit $\sigma\to\infty$, where $\sigma+1$ is the coordination number of the Cayley tree.

(2) We note that (a) from the integral equation we calculate the density of states for the model with a Gaussian distribution of hopping matrix elements in the limit $1\ll\sigma\ll\infty$ in order to show the nonsingular behavior of the density of states around $E=E_c$. Also, it is shown that around $E=E_c$ the dominant contribution to the density of states comes from the coupling between the perturbative and the nonperturbative saddle-point contributions. We believe the latter correspond to instanton contributions for the hypercubic lattices. For $E^2\gg E_c^2$ the dominant contribution comes from the saddle point integral only. (b) At large $|E|$ we obtain the result

$$\rho(E)=CE^{\sigma}\exp(-E^2/2)\ , \tag{5.3}$$

which was previously derived by a variational argument.[16]

(3) For large but finite σ we find that the singularity in g which leads to the result in Eq. (5.2) is shifted off the real axis, so that g has a square-root singularity at an energy near E_c, but has an imaginary part $\mathrm{Im}E\sim\exp(-\sigma/2)$. This singularity is on the unphysical energy sheet of g and does not give rise to singular behavior in $\rho(E)$ on the real axis.

(4) For the model with a bounded distribution, using the Lifschitz argument, we have calculated the density of states near the band edge.

(5) We also suggest a description of band formation using the language of dynamical systems. Our work supports the conclusion that as far as the density of states is concerned there is no sharp transition inside the band.

ACKNOWLEDGMENTS

We would like to thank Dr. T. C. Lubensky for helpful suggestions. We also gratefully acknowledge support for that part of the work done at the University of Pennsylvania from the National Science Foundation under Grant No. DMR-82-19216, and from the Office of Naval Research under Grant No. ONR-0158.

APPENDIX A: PERTURBATIVE SOLUTION TO EQS. (2.36) AND (3.8)

In this appendix we discuss the solution to the equation

$$\Lambda(a)=a^2-a+\frac{1}{e^2}+\frac{\phi(a,e)}{e^2}=0\ , \tag{A1}$$

where ϕ is considered a perturbation and a is the unknown. In general, a perturbative solution to Eq. (A1) can be obtained by elementary methods. The only difficulty here is that we wish to obtain a perturbative solution which is valid near the singular point where $d\Lambda(a)/da=0$, and perturbation theory is nontrivial.

To analyze Eq. (A1) we expand about the unperturbed solution, which we denote a_0:

$$a^2-a+\frac{1}{e^2}+\frac{1}{e^2}\left[\phi(a_0,e)+(a-a_0)\frac{\partial\phi(a_0,e)}{\partial a}\right]=0\ . \tag{A2}$$

$\partial\phi/\partial a$ will not appear in the final answer, but it is useful to retain it to show the appropriateness of the expansion we develop. Solving the quadratic in Eq. (A2), we find

$$2a=1-\frac{1}{e^2}\frac{\partial\phi}{\partial a}-\left[1-\frac{\Omega(e)}{e^2}\right]^{1/2} \tag{A3}$$

to lowest order in ϕ, where

$$\Omega(e)=4+4\phi(a_0(e),e)-[4a_0(e)-2]\frac{\partial\phi(a,e)}{\partial a}\bigg|_{a=a_0(e)}\ . \tag{A4}$$

One might expect $\Omega(e)$ to have a square-root singularity in e for e near $e^2=4\equiv e_c^2$ in view of the singularity in a_0. However, expanding $\Omega(e)$ about $e^2=e_c^2$ shows that such a singularity is not present.

The complex value of the singularity in a is located by

$$e_c^2=\Omega(e_c)\ , \tag{A5}$$

so that

$$2a=1-\frac{1}{e^2}\frac{\partial\phi}{\partial a}-\left[1-\frac{e_c^2}{e^2}-\frac{\Omega(e)-\Omega(e_c)}{e^2}\right]^{1/2}\ . \tag{A6}$$

Since, as mentioned, $\Omega(e)$ has no square-root singularity, we may expand Eq. (A6) in powers of Ω so that

$$2a=1-\frac{1}{e^2}\frac{\partial\phi}{\partial a}-(1-e_c^2/e^2)^{1/2}-\frac{\Omega(e)-\Omega(e_c)}{2e^2(1-e_c^2/e^2)^{1/2}} \tag{A7a}$$

$$=1-\left[(1-e_c^2/e^2)^{1/2}+\frac{2[\phi(e)-\phi(e_c)]}{e^2(1-e_c^2/e^2)^{1/2}}\right]\ , \tag{A7b}$$

where $\phi(e)=\phi(a_0(e),e)$, and Eq. (A5) can be written as

$$e_c^2=4\phi(a_0(e_c),e_c)+4\ . \tag{A8}$$

In Eq. (A8) we have accomplished our objective. This expansion has the correct analytic properties to preserve the square-root singularity in the complex e plane, but is valid to lowest order in ϕ for all e. From this equation we can now easily generate explicit results for both $e \sim e_c$ and $e \gg e_c$.

Equation (2.36) is in the same form as Eq. (A1), and thus the perturbative solution of Eq. (2.36) is from Eqs. (A7b) and (A8),

$$a_1 = a_1^{(0)} + (1/\sigma) a_1^{(1)} , \tag{A9}$$

with

$$a_1^{(0)} = \frac{1}{2} - \frac{1}{2} \left[\frac{E^2 - E_c^2}{E^2} \right]^{1/2} \tag{A10a}$$

and

$$a_1^{(1)} = \frac{\delta_t}{[t^2]_t} \left[1 - 2\frac{E^2}{E_c^2} + 2\frac{E^2}{E_c^2} \left[\frac{E^2 - E_c^2}{E^2} \right]^{1/2} \right] , \tag{A10b}$$

where

$$E_c^2 = 4\sigma [t^2]_t \left[1 + \frac{1}{\sigma} \frac{\delta_t}{[t^2]_t} \right] . \tag{A11}$$

Then, from the formula (2.18), the Green's function to order σ^{-1} is given by

$$Eg(E) = \frac{1}{1-a_1} + \frac{1}{\sigma} \left[\frac{2a_2}{(1-a_1)^3} + \frac{a_1}{(1-a_1)^2} \right] . \tag{A12}$$

Therefore, from Eqs. (A9) and (A12) we derive Eqs. (2.38) and (2.39).

APPENDIX B: DETAILED CALCULATIONS FOR $\rho_>(E)$

In this appendix, using the results of Sec. III A and Appendix A, we give the detailed calculations for the density of states for $E^2 > E_c^2 = (\mathrm{Re}E_1)^2$ in the limit $\sigma \to \infty$ (see Sec. III B).

The self-consistent equation (3.8) is of the form of Eq. (A1). Therefore, using Eqs. (A7) and (A8), one can write

$$a_1 = a_1^{(0)} + \frac{1}{\sigma} a_1^{(1)} - i a_1^{(\gamma)} , \tag{B1}$$

where

$$a_1^{(0)} = \frac{1}{2} - \frac{1}{2} \left[1 - \frac{E_1^2}{E^2} \right]^{1/2} , \tag{B2a}$$

$$a_1^{(1)} = 2 \left[1 - 2\frac{E^2}{E_c^2} + 2\frac{E^2}{E_c^2} \left[1 - \frac{E_1^2}{E^2} \right]^{1/2} \right] , \tag{B2b}$$

and

$$e^2 a_1^{(\gamma)} = \chi(e) + \left[1 - \frac{E_1^2}{E^2} \right]^{-1/2} [(a_1^{(0)})^{-1/2} \chi(e) - \sqrt{2}\chi(e_c)] , \tag{B2c}$$

with

$$\chi(e) = i \int_{C_2} \exp[-\psi(s)] ds \tag{B3a}$$

and

$$e_1^2 = e_c^2 - 4\sqrt{2} i \chi(e_c) , \tag{B3b}$$

where

$$E_c^2 = \sigma e_c^2 = 4\sigma(1 + 2/\sigma) . \tag{B3c}$$

Equation (B3b) is equivalent to Eq. (3.15). To calculate the Green's function, we should evaluate the integral

$$g(E) = \frac{1}{E} \int_0^\infty e^{-u} [\phi(u)]^{\sigma+1} du . \tag{B4}$$

In order to use the saddle-point method, the integral (B4) should be divided into two parts,

$$g(E) = \frac{1}{E} \int_{C_1'} e^{-u} [\phi(u)]^{\sigma+1} du + \frac{1}{E} \int_{C_2'} e^{-u} [\phi(u)]^{\sigma+1} du$$
$$= I_{C_1'} + I_{C_2'} , \tag{B5}$$

where the contours C_1' and C_2' are similar to, but slightly different from, the ones depicted in Fig. 3. The first part of the integrals (B5) is from Eq. (A12),

$$I_{C_1'} = \frac{1}{E(1-a_1^{(0)})} + \frac{1}{\sigma E} \left[\frac{a_1^{(1)}(1-a_1^{(0)}) + a_1^{(0)}(1+a_1^{(0)})}{(1-a_1^{(0)})^3} \right] - i \frac{a_i^{(\gamma)}}{E(1-a_1^{(0)})^2} , \tag{B6}$$

and thus the density of states in the region $E^2 > E_c^2$ is

$$\rho_>(E) = -\frac{\mathrm{Im}a_1^{(0)}}{\pi E(1-\langle a_1^{(0)} \rangle)^2} + \frac{\mathrm{Re}a_1^{(\gamma)}}{\pi E(1-\langle a_1^{(0)} \rangle)^2} - \frac{1}{\pi} \mathrm{Im} I_{C_2'} , \tag{B7}$$

where $\langle a_1^{(0)} \rangle = \mathrm{Re}a_1^{(0)}$, and

$$\mathrm{Im}a_1^{(0)} = -\frac{1}{2} \left[\left[1 - \frac{E_c^2}{E^2} \right]^2 + \frac{32\sigma^2 \chi(e_c)^2}{E^4} \right]^{1/4} \sin \left[\tfrac{1}{2} \tan^{-1} \left[\frac{4\sqrt{2}\sigma\chi(e_c)}{E^2 - E_c^2} \right] \right] . \tag{B8}$$

Using $\phi(u)$ in the relation (3.3) as a leading-order approximation, $\rho_>(E)$ is

$$\rho_>(E) = \rho_>^{(c)}(E) + \frac{4\sigma\chi(E)}{\pi[Q(E)]^2 E^3} \{1 + (E^2/\sigma)^{3/2} [Q(E)/2]^{5/2} \exp(\tfrac{1}{2})\} \tag{B9}$$

where

$$\rho_{>}^{(c)}(E)=\frac{2}{\pi E}[Q(E)]^{-2}\left[\left[1-\frac{E_c^2}{E^2}\right]^2+\frac{32\sigma^2\chi^2(E_c)}{E^4}\right]^{1/4}\sin\left[\tfrac{1}{2}\tan^{-1}\frac{4\sqrt{2}\sigma\chi(E_c)}{E^2-E_c^2}\right]$$
$$+\frac{1}{\pi E}[Q(E)/2]^{-2}\left[\left[1-\frac{E_c^2}{E^2}\right]^2+\frac{32\sigma^2\chi^2(E_c)}{E^4}\right]^{1/4}\cos\left[\tfrac{1}{2}\tan^{-1}\frac{4\sqrt{2}\sigma\chi(E_c)}{E^2-E_c^2}\right]$$
$$\times\left[\left[\frac{\sigma Q(E)}{2E^2}\right]^{1/2}\chi(E)-\frac{\sqrt{2}\sigma}{e^2}\chi(E_c)\right] \tag{B10a}$$

and

$$\chi(E)=\tfrac{1}{2}(\pi\sigma)^{1/2}(E^2/\sigma)^{\sigma/2}[Q(E)/2]^{\sigma/2}\exp\left\{-\frac{E^2}{8}\left[1+\left[1-\frac{E_c^2}{E^2}\right]^{1/2}\right]^2\right\}, \tag{B10b}$$

$$Q(E)=1+\left[1-\frac{E_c^2}{E^2}\right]^{1/2}. \tag{B10c}$$

As can be seen from Eq. (B7), the term $\rho_{>}^{(c)}(E)$ and the contribution proportional to $\chi(E)E^{-3}$ in formula (B9) are due to the coupling between the perturbative correction (see Sec. II C) and the nonperturbative correction (i.e., the steepest-descent integral) through the coefficient a_1. The contribution proportional to $\chi(E)$ is due to the direct nonperturbative correction to the integral for the Green's function $g(E)$. Also, as one sees from Eq. (B10a), $\rho_{>}^{(c)}$ has crossover behavior from $\rho_{>}{}^{(c)}\sim\chi(E)$ for $E^2\gg E_c^2$ to $\rho_{>}^{(c)}\approx[\chi(E_c)]^{1/2}$ for $E^2\approx E_c^2$. Here, then, the dominant contribution to the density of states comes from the coupling between the perturbative correction and nonperturbative correction. However, in the limit $E^2\to\infty$ the dominant contribution comes from the direct nonperturbative correction, i.e., through the term proportional to $\chi(E)$. The energy δE around E^2 which defines this crossover regime is given by (3.19).

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